Solving the Chapman-Kolmogorov equation for a jumping process

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A general solution to the Chapman-Kolmogorov equation for a jumping process called the "kangaroo process" is derived. A special case of algebraic dependences is discussed in detail. In particular, simple asymptotic formulas for probability distribution are presented. It is demonstrated that there are two different classes of limiting stationary distributions. An expression for the covariance is also derived.

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I. INTRODUCTION

It became clear in recent years that a stochastic description of many physical phenomena requires a more general class of stochastic processes than the white noise, possessing no correlations and being Gaussianly distributed. Long-time, algebraic covariances appear in the fluid dynamics [1–3], linearized hydrodynamics [4], for the noise-induced Stark broadening phenomenon [5]. Moreover, if one tries to describe a complicated dynamical system in terms of a single, effective variable and some noise responsible for fluctuations, that noise must be correlated [6,7]. The Brownian particle trajectory is given in this case by the generalized Langevin equation in which the stochastic force must be represented by a process possessing final correlation time. On the other hand, there are strong indications of non-Gaussian noises appearances in nature [8,9]; the algebraically distributed noise serves to generate Lévy flights as a driving force in the Langevin equation [10]. Similarly, the fractional Fokker-Planck equation corresponds to nonlinear Langevin equation driven by non-Gaussian noise [11].

Emergence of these phenomena emphasizes the importance of stochastic processes that possess a general form of covariance and arbitrary distribution shape, not necessarily Gaussian. Such processes can still be Markovian and satisfy the forward Chapman-Kolmogorov equation (CKE) [12]:

$$\frac{\partial}{\partial t}p(x,t)=-\frac{\partial}{\partial x}[A(x,t)p(x,t)]+\frac{\partial^2}{\partial x^2}[B(x,t)p(x,t)]+\int dx'[W(x|x',t)p(x',t)-W(x'|x,t)p(x,t)], \tag{1}$$

where coefficients A, B, and W are defined in terms of a given transition probability $p_{tr}(x,t|x_0,t_0)$. In particular,

$$W(x|x',t)=\lim_{\Delta t\to 0} p_{tr}(x,t+\Delta t|x',t)/\Delta t. \tag{2}$$

If we assume that $W(x|y,t)$ vanishes, Eq. (1) resolves itself to the general diffusion equation describing a process with continuous paths. Its solutions are restricted to the Gaussians, but long-time correlations are not excluded [13–16]. The integral term, in turn, is responsible for jumping processes and it can produce probability distributions with various shapes and possessing various forms of the covariance.

In this paper, we consider a special form of the jumping process called the kangaroo process (KP) [17]. It is a stepwise constant Markov process defined for infinitesimal time intervals by the following stationary transition probability:

$$p_{tr}(x,\Delta t|,x',0)=[1-\nu(x')\Delta t]\delta(x'-x)+\nu(x')\Delta t Q(x), \tag{3}$$

where $Q(x)$ is the given probability density and $\nu(x)$ is the jump frequency. Therefore, $p_{tr}dx$ means the probability that KP value is between x and $x+dx$ at time Δt, knowing that it was equal to x' at time 0. The first term on the right-hand side of Eq. (3) is the probability that no jump occurred in the time interval $(0,\Delta t)$. The term $\nu(x')\Delta t$ means the probability that one jump occurred. Directly after such a jump, the probability density of x becomes $Q(x)$. To get the CKE for KP, we insert the transition probability (3) into Eq. (1). One can verify that both coefficients A and B vanish and CKE is fully determined by the integral component. It takes the form

$$\frac{\partial}{\partial t}p(x,t)=-\nu(x)p(x,t)+Q(x)\int \nu(x')p(x',t)dx'. \tag{4}$$

KP is a stationary Markov process; it is characterized by some stationary probability distribution, not necessarily Gaussian, and it can possess an arbitrary covariance. In addition, there exists a simple procedure allowing us to construct a KP if those quantities are given *a priori* [17]. This fact makes the KP very useful for applications, e.g., as a model of the stochastic force in the generalized Langevin equation [18–20]. In the framework of kinetic theories, KP serves as a model of isotropic collision kernel in the linear Boltzmann equation [21]. Moreover, KP can describe a turbulent, nondiffusive transport process in fluids [22].

Many applications of KP are restricted to stationary solutions. Those applications that regard KP as a model of some physical phenomenon rely on a tacit assumption that the limiting distribution can be reached fast for almost any initial condition, what generally is justified only for colored noises (exponentially falling correlations). Since, recently, the importance of other kinds of noise has been widely recognized, the KP requires deeper insight and more detailed elaboration. In this paper we derive the general, time-dependent solution of CKE for KP. We demonstrate how KP should be handled, in particular, with respect to the choice of initial conditions, to make the convergence as fast as possible. Another motivation for using stationary solutions is that they are simple and very easy to obtain. We show that one can get very simple expressions for time-dependent solutions as well.

The details of the derivation of CKE solution and discussion of the general case are presented in Sec. II. In Sec. III we consider the case for which the functions defining the process, $Q(x)$ and $\nu(x)$, depend algebraically on x and we construct asymptotic solutions for large t. Fluctuation characteristics of the process—covariance and variance—are derived in Sec. IV. The results are summarized and discussed in Sec. V.

II. GENERAL SOLUTION OF THE CKE FOR KP

The stationary solution of Eq. (4) is given by

$$P_1(x)=\frac{Q(x)\langle\nu(x)\rangle}{\nu(x)}=\frac{Q(x)/\nu(x)}{\int Q(x')/\nu(x')dx'}. \tag{5}$$

Clearly, this solution is valid only for such $\nu(x)$ and $Q(x)$ that the distribution $P_1(x)$ is normalizable, i.e., the integral in Eq. (5) is finite. We call the solution $P_1(x)$ "normal." The normalization condition may not be satisfied if frequency $\nu(x)$ equals zero for some $x=a$, which implies the existence of infinitely long free paths. This case is physically very important; infinite jumps are necessary for Markovian processes to possess long tails in the covariance function. Certainly, for such a choice of $\nu(x)$ and $Q(x)$ that $\int P_1(x)dx=\infty$, the solution of Eq. (4) does not converge to the normal stationary distribution $P_1(x)$. However, in this case another stationary distribution emerges:

$$P_2(x)=\delta(x-a), \tag{6}$$

which we call the singular stationary distribution. In general, an actual stationary solution can comprise both types of distribution.

In order to find the general, time-dependent solution, we apply the integral transform technique. We take the Laplace transform of both sides of Eq. (4) with respect to time t. Denoting the transform of $p(x,t)$ by $G(x,s)$, $\mathcal{L}_t(p(x,t))=G(x,s)$, the solution of the transformed equation can be cast into the following form:

$$G(x,s)=\frac{Q(x)C_s}{s+\nu(x)}+\frac{p(x,0)}{s+\nu(x)}, \tag{7}$$

where $p(x,0)$ is an initial distribution and $C_s=\int\nu(x)G(x,s)dx$ depends only on s. We can then calculate C_s just by inserting into the above integral $G(x,s)$ from Eq. (7). The final solution of the transformed CKE reads

$$G(x,s)=\frac{Q(x)\int \nu(x')p(x',0)/[s+\nu(x')]dx'}{[s+\nu(x)]s\int Q(x')/[s+\nu(x')]dx'}+\frac{p(x,0)}{s+\nu(x)}. \tag{8}$$

Our aim is to invert the function $G(x,s)$. The main difficulty in achieving that is connected with the integral in the denominator; the other terms can be easily inverted. Therefore, we separate the term with the integral and express the final solution as the following convolution:

$$p(x,t)=p(x,0)\exp[-\nu(x)t]+Q(x)\int_0^t\int \nu(x')p(x',0)\times\exp[-\nu(x')\tau]dm'\mathcal{L}_\tau^{-1}\times\left(\frac{1}{s[s+\nu(x)]\left\{\int Q(x')/[s+\nu(x')]dx'\right\}}\right)d\tau. \tag{9}$$

The inverse transform in Eq. (9) does exist, because the original function tends to zero with $s\to\infty$. We perform inversion by applying the usual formula

$$\mathcal{L}_t^{-1}[f(s)]=\frac{1}{2\pi i}\int_{-i\infty+\sigma}^{i\infty+\sigma}f(s)\exp(st)ds. \tag{10}$$

The integrand is an analytic function; it possesses two simple poles, at $s=0$ and at $s=-\nu(x)$, as well as two branch points x_1 and x_2, which are real and nonpositive. The contour of integration then comprises small circles around those points and straight line segments along the real negative half-axis, on both sides of cut. The result of integration gives us the inverse Laplace transform that appears in Eq. (9). It depends explicitly on stationary solution, either normal or singular. The transform reads

$$\mathcal{L}_t^{-1}=-P_i(x)/Q(x)-\frac{\exp(-\nu(x)t)}{\nu(x)\int Q(x')/[\nu(x')-\nu(x)]dx'}-\mathrm{P}\int_{x_1}^{x_2}\frac{\exp(x't)}{x'[x'-\nu(x)]}g(x')dx'\quad(i=1,2). \tag{11}$$

The auxiliary function $g(x)$ estimates the difference of the integrand values on different branches:

$$g(x)=\lim_{\epsilon\to0^+}\left(\frac{1}{\int Q(x')/[x+i\epsilon+\nu(x')]dx'}-\frac{1}{\int Q(x')/[x-i\epsilon+\nu(x')]dx'}\right). \tag{12}$$

Performing the convolution produces the final result:

$$p(x,t)=\overbrace{p(x,0)\exp[-\nu(x)t]}^{I}+\overbrace{P_i(x)}^{II}-\overbrace{P_i(x)\int p(x',0)\exp[-\nu(x')t]dx'}^{III}$$
$$+\overbrace{\frac{Q(x)\exp[-\nu(x)t]}{\nu(x)\int Q(x')/[\nu(x')-\nu(x)]dx'}\int\frac{\nu(x')p(x',0)}{\nu(x')-\nu(x)}dx'}^{IV}+\overbrace{\frac{Q(x)}{\nu(x)\int\frac{Q(x')}{\nu(x')-\nu(x)}dx'}\int\frac{\nu(x')p(x',0)}{\nu(x')-\nu(x)}\exp[-\nu(x')t]dx'}^{V}$$
$$+\overbrace{Q(x)P.V.\int_{x_1}^{x_2}dx''\int\frac{\nu(x')g(x'')p(x',0)}{x''[x''+\nu(x)][x''-\nu(x')]}\{\exp(x''t)-\exp[-\nu(x')t]\}dx'}^{VI}\qquad(i=1,2). \tag{13}$$

The most interesting is the asymptotic behavior of $p(x,t)$ at $t\to\infty$. We can then simplify Eq. (13), preserving only those terms that fall down with time sufficiently slowly. The relative importance of subsequent terms depends on whether we deal with the normal or with the singular case. One can demonstrate that in the normal case the term VI drops with time faster than III and V and, therefore, it can be neglected. On the other hand, the term VI becomes important for the singular case, in contrast with III, IV, and V, which, in turn, can be dropped. Finally, the term I falls exponentially in both cases and it can be neglected, unless $p(x,0)=\delta(x)$.

The choice of the initial condition $p(x,0)$ is essential for the actual solution of the CKE in the asymptotic regime. In particular, this function decisively influences the speed of convergence to the stationary state. Let us consider from that point of view the integral in term III. It can be comprehended as the Laplace integral: $f(t)\equiv\int p[x,0)\exp(-\nu(x)t](dx/d\nu)d\nu$. Therefore, we can evaluate an initial condition $p(x,0)$, which leads to some *a priori* assumed $f(t)$, with a given $\nu(x)$, just by inverting the Laplace transform: $p(x,0)=\mathcal{L}_\nu^{-1}[f(t)](d\nu/dx)$.

Equation (13) determines the general solution of CKE for KP for any given functions $Q(x)$ and $\nu(x)$. Depending on a particular form of these functions, as well as on the initial condition, the solution converges to the normal stationary state $P_1(x)$, the singular one $P_2(x)$ or to a combination of both. For some cases, the convergence is so slow that the process itself can hardly be regarded as stationary. We present this problem in detail for algebraic dependences of $Q(x)$ and $\nu(x)$ in the following section.

III. APPLICATION TO ALGEBRAIC DEPENDENCES

Probability distributions possessing power-law tails are distinguished because they are characterized by divergent moments. As a consequence, stochastic trajectories exhibit long jumps, frequently observed in physical phenomena. In this section we consider the KP defined by $Q(x)$ and $\nu(x)$, which depend algebraically on x. The resulting probability distributions are also algebraic at large time.

A. Normal stationary asymptotic solution

Let the stochastic variable x be defined on the interval $[-1,1]$ and the process itself by the following functions:

$$Q(x)=|x|^\alpha,\quad \nu(x)=|x|^\beta. \tag{14}$$

In addition, we assume the homogeneous initial condition $p(x,0)=0.5$. The requirement of the existence of a normal stationary state, $P_1(x)=[Q(x)/\nu(x)]/\int Q(x')/\nu(x')dx'=0.5(\alpha-\beta+1)|x|^{\alpha-\beta}$, imposes the following condition on the parameters: $\beta<\alpha+1$. The solution of the CKE (13) reads in this case as

$$p(x,t)\approx P_1(x)\left(1+\int_0^1\exp(-t|x'|^\beta)dx'\right)+\frac{|x|^{\alpha-\beta}}{2\int_0^1|x'|^\alpha/(|x'|^\beta-|x|^\beta)dx'}\times\int_0^1\frac{|x'|^\beta\exp(-t|x'|^\beta)}{|x'|^\beta-|x|^\beta}dx'. \tag{15}$$

Since we are interested in the asymptotic behavior $t\to\infty$, the terms that fall down quickly with time have been neglected in Eq. (15). The above equation can still be simplified. Due to the exponential factor in the integrands, in the limit $t\to\infty$, only very small values of the integration variable contribute to the integral, providing $x\neq0$ [23]. Therefore, the first integral can be estimated in the following way: $\int_0^1\exp(-t|x'|^\beta)dx'\approx\int_0^\infty\exp(-t|x'|^\beta)dx'\sim t^{-1/\beta}$. The second integral, in turn, is proportional to the derivative of the first one: it falls like $t^{-1-1/\beta}$ and can be neglected. The final asymptotic formula for the probability distribution is then

$$p(x,t)\approx P_1(x)[1-t^{-1/\beta}\Gamma(1/\beta)/\beta]\quad(x\neq0). \tag{16}$$

The solution for $x=0$ ($\beta>0$) can be obtained directly from Eq. (4): $dp(0,t)/dt=0$, then $p(0,t)=\text{const}=p(0,0)$. This finding contradicts the stationary asymptotic solution $P_1(0)$,

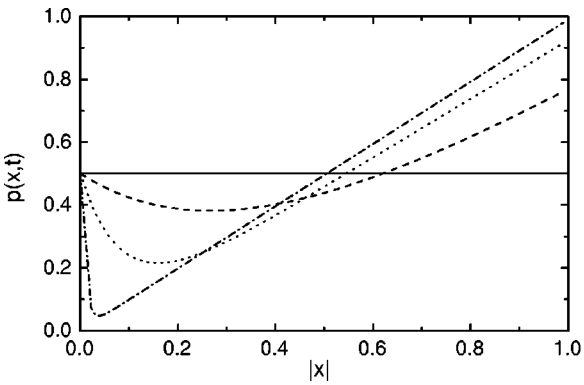

FIG. 1. Probability distribution $p(x,t)$ for the normal case, calculated according to (17), as a function of $|x|$ at times $t=0$ (solid line), $t=2$ (long dashes), $t=10$ (short dashes), and $t=100$ (dash-dotted line).

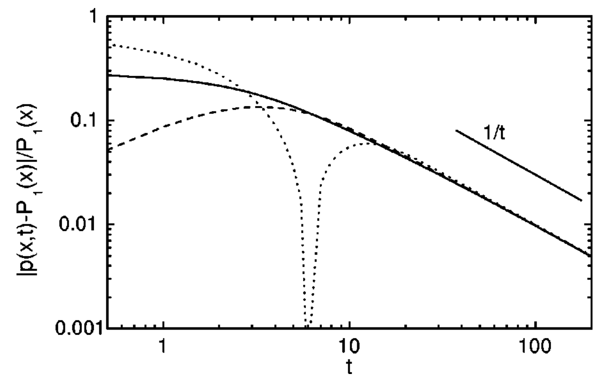

FIG. 2. Relative deviation of the probability distribution (17) from the stationary limiting distribution $P_1(x)$ as a function of time. Results for the following values of x are presented: $x=0.7$ (solid line), $x=0.5$ (long dashes), and $x=0.3$ (short dashes). For comparison, the slope $1/t$ is shown. The singularity for the $x=0.3$ case corresponds to the time value for which the plotted function passes through zero.

which equals zero if $\alpha>\beta$. Therefore, the distribution the system converges to, must be discontinuous at $x=0$.

In order to illustrate the above formulas, let us consider a simple example: $\alpha=2$ and $\beta=1$. The exact result, involving all the terms in Eq. (13), reads

$$p(x,t) = \frac{1}{2}\exp(-|x|t)+|x|-\frac{x^2}{4}\Bigg[\exp(-|x|t)A(x) + \int_0^1 \frac{\exp(-x't)dx'}{(|x|-x')\{[0.5+x'+x'^2\ln(1/x'-1)]^2+\pi^2x'^4\}}\Bigg], \tag{17}$$

where

$$A(x)=\frac{|x|[\ln a+2(|x|\ln a+1)^2+2\pi^2|x|^2]+1}{(0.5+|x|+|x|^2\ln a)^2+\pi^2|x|^4},$$

$$a=1/|x|-1. \tag{18}$$

Figure 1 demonstrates how this distribution evolves with time. The straight line corresponding to the stationary limit $P_1(x)=|x|$ is reached at relatively short times if $|x|$ is not very small. In the vicinity of the point $x=0$, the curve rapidly bends upwards to the value $p(0,t)=0.5$.

Figure 2 shows that the asymptotic time dependence $1/t$, resulting from expression (16), reproduces the exact result quite precisely even at short times. Moreover, the plotted quantity appears to be asymptotically x independent, in agreement with Eq. (16).

B. Singular stationary asymptotic solution

If conditions for parameters α and β from the preceding section are not met, the normal stationary state $P_1(x)$ no longer exists. Instead, the process is expected to converge to the singular state $P_2(x)$. We consider the following choice of functions Q and ν:

$$Q(x)=|x|^\alpha,\quad \nu(x)=|x|^{\alpha+1}, \tag{19}$$

where $x\in[-1,1]$. The corresponding stationary solution is then: $P_2(x)=\delta(x)$. We assume the same initial condition as for the normal case: $p(x,0)=0.5$. Now, the most important term in Eq. (13) is VI. It takes the form

$$p(x,t)\approx-\frac{\exp(-|x|t)\ln(1/|x|-1)}{|x|^{\alpha+1}[\ln^2(1/|x|-1)+\pi^2]}+(\alpha+1)|x|^\alpha \times\int_0^1\frac{\exp(-x't)}{x'[|x|^{\alpha+1}-x'][\ln^2(1/x'-1)+\pi^2]}dx'. \tag{20}$$

We want to find the asymptotic expression for Eq. (20), first for $x\neq0$. As for the normal stationary case, we try to simplify integral (20) utilizing the fact that for large t only small values of the integration variable contribute to the integral. Therefore, $p(x,t)\approx(\alpha+1)/|x|\int_0^a\exp(-tx')/x'\ln^2x'dx'$, where $a\ll1$. We estimate the value of the above integral by approximating the exponent from below by a straight line segment: $\exp(-tx)\geqslant1-tx$, where $x\in(0,1/t)$. Therefore,

$$\int_0^a\frac{\exp(-tx)}{x\ln^2x}dx\geqslant\int_0^{1/t}\frac{1-tx}{x\ln^2x}dx. \tag{21}$$

The leading term at $t\to\infty$ in the right-hand side integral behaves like $1/\ln t$. To estimate the value of our integral from above, we choose some point $x_t\in(0,\,a)$ and approximate the exponent by a function $g_t(x)$, defined as two straight line segments that coincide with the exponential function at three points: $x=0$, x_t, and a. Therefore,

$$\int_0^a\frac{\exp(-tx)}{x\ln^2x}dx\leqslant\int_0^a\frac{g_t(x)}{x\ln^2x}dx. \tag{22}$$

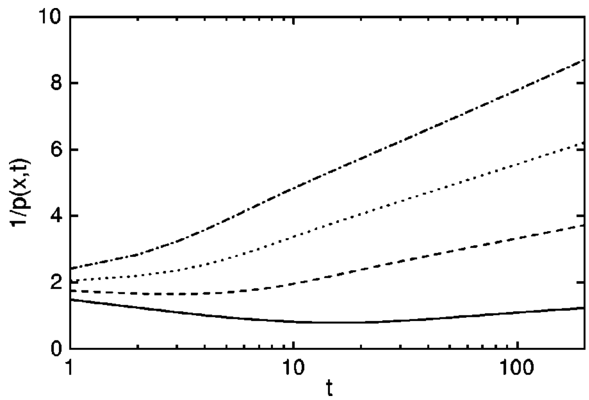

FIG. 3. Reciprocal of probability distribution $p(x,t)$ for the singular case (19), with $\alpha=0$, calculated according to Eq. (20), as a function of time. Curves in the figure correspond to the following process values: $x=0.1$ (solid line), $x=0.3$ (long dashes), $x=0.5$ (short dashes), and $x=0.7$ (dash-dotted line).

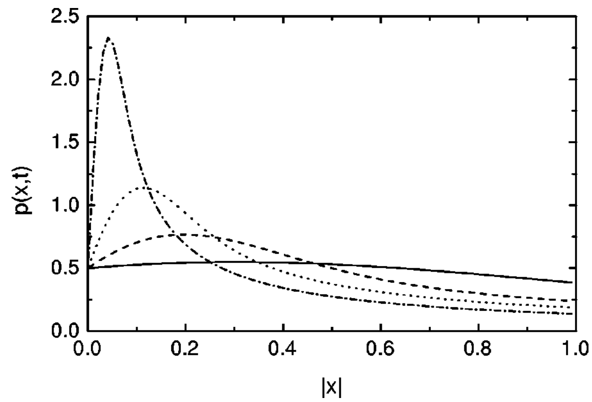

FIG. 4. Exact probability distribution $p(x,t)$ for the singular case (19) with $\alpha=1$, evaluated from Eq. (13), for the times $t=1$ (solid line), $t=10$ (long dashes), $t=50$ (short dashes), and $t=500$ (dash-dotted line).

We assume time dependence of the point x_t in the form $x_t=t^{-\gamma}$, where $0<\gamma<1$. The corresponding slope of the left segment for large t equals $-t^{\gamma}$, $g_t(x)\approx 1-t^{\gamma}x$, and this segment contributes predominantly to the integral. Then in the limit $\gamma\to 1^-$, both approximations of the exponential function converge. Evaluating the right-hand side integral in Eq. (22) and taking the limit $\gamma\to 1^-$, we find that the leading term in the resulting expression coincides with the outcome previously obtained as estimation from below: $1/\ln t$. Then the final asymptotic formula for the solution in the singular case reads

$$p(x,t)\approx\frac{\alpha+1}{|x|\ln t}\quad (x\neq 0). \tag{23}$$

Note that the shape of the distribution, with respect either to x or to t, does not depend on α.

The asymptotic probability distribution for $x=0$ can be obtained similarly. One should evaluate the limit $x\to 0$ in Eq. (20) and then estimate the resulting integral both from below and from above. We get the simple result

$$p(0,t)\approx t/\ln t. \tag{24}$$

As an example, let us consider the $\alpha=0$ case. Since the terms I–V in Eq. (13) vanish in this case, Eq. (20) gives us the exact solution, the reciprocal of which is presented in Fig. 3 in the semilogarithmic scale. This form of the plot clearly demonstrates the inverse logarithmic time dependence of distribution (20)—the curve becomes a straight line for large t—in accordance with Eq. (23). This asymptotic dependence is reached at short time if $|x|$ is close to 1.

Processes with $\alpha>0$ possess asymptotically the same distribution (23) as the exemplary case considered above. However, there is a difference at the point $x=0$, where the solution resembles the normal case (see Fig. 1): inserting, e.g., $x=0$ to the original CKE (4), we obtain the straightforward solution $p(0,t)=\text{const}=p(0,0)$. Therefore, the distribution $p(x,t)$ rises with time to infinity in the closest neighborhood of $x=0$, but at this point itself it must be kept at a constant value. Figure 4 shows how the exact distribution $p(x,t)$, evaluated according to Eq. (13) for $\alpha=1$, approaches that extraordinary shape.

IV. COVARIANCE

An interesting feature of KP, and important one from the point of view of applications, is that it possesses a very broad class of possible covariance functions. One can construct the process for almost arbitrary form of covariance by changing the $Q(x)$ and $\nu(x)$ functions.

The covariance is defined by the following average:

$$\mathcal{C}(t_0,\tau)=\langle x(t_0)x(t_0+\tau)\rangle, \tag{25}$$

performed over the CKE solution $p(x,t)$, where $\tau=t_1-t_0$ is a time increment. The covariance can be expressed in terms of the conditional probability $P(x,\tau|x')$ [12] by integrating over the CKE solution $p(x,t)$, taken at the final time $t=t_1$:

$$\mathcal{C}(t_1,\tau)=\int\int x'(t_1-\tau)x(t_1)P(x,\tau|x')p(x,t_1)dxdx'. \tag{26}$$

The evaluation of conditional probability $P(x,\tau|x')$ is presented in the Appendix. Its Laplace transform takes the form

$$\mathcal{L}_\tau[P(x,\tau|x')]=\frac{\delta(x'-x)}{s+\nu(x')}+\frac{\nu(x')Q(x)}{[s+\nu(x')][s+\nu(x)]}\times\frac{1}{s\int\dfrac{Q(x'')}{s+\nu(x'')}dx''}. \tag{27}$$

Then, we take Laplace transform of Eq. (26) with respect to τ, $\mathcal{L}_\tau[\mathcal{C}(t,\tau)]=\tilde{\mathcal{C}}(t,s)$, and apply expression (27) for the transform of the conditional probability. After performing the integrals, we finally obtain

$$\tilde{\mathcal{C}}(t_1,s)=\int x'^2 p(x',t_1)/[s+\nu(x')]dx'+\frac{\int x'\nu(x')p(x',t_1)/[s+\nu(x')]dx'\int xQ(x)/[s+\nu(x)]dx}{s\int Q(x'')/[s+\nu(x'')]dx''}. \tag{28}$$

The above expression becomes especially simple if $p(x,t)$ as well as $\nu(x)$ are even functions of variable x. In that case the second component of Eq. (28) vanishes. The remaining term can be easily inverted to the form

$$\mathcal{C}(t_0,\tau)=\int x^2\exp[-\nu(x)\tau]p(x,t_0+\tau)dx. \tag{29}$$

One can utilize Eq. (29) to find the covariance for both the cases considered in the preceding section. Unfortunately, the asymptotic expressions for $p(x,t)$ cannot be applied because approximations used to derive them are poor for very small x. Since we are interested predominantly in large τ, small values of x are important: in fact they constitute the main contribution to integral (29). Consequently, the evaluation of the covariance function tail requires taking into account the complete expression for $p(x,t)$, given by Eq. (13).

On the other hand, the covariance for the normal case in its stationary limit, characterized by the probability distribution $P_1(x)$, can be easily evaluated. In particular, for KP defined by Eq. (14), it reads

$$\mathcal{C}(\tau)=\frac{2}{\beta(\alpha-\beta+1)}\Gamma\left(\frac{\alpha-\beta+3}{\beta}\right)\tau^{-2/\beta}. \tag{30}$$

Brissaud and Frisch [17] demonstrated that there exists some stationary KP for any choice of both covariance $\mathcal{C}(\tau)$ and distribution $P_1(x)$. It can be easily constructed by solving a simple differential equation.

In contrast to the general formula for the covariance, the variance of the process, $\sigma^2=\langle x^2(t_0)\rangle$, can be determined by using asymptotic expressions for $p(x,t)$; we need only to insert $\tau=0$ into Eq. (29). In the normal case, we have

$$\sigma^2(t_0)=\frac{2}{(\alpha-\beta+1)(\alpha-\beta+2)}[1-t_0^{-1/\beta}\Gamma(1/\beta)/\beta]. \tag{31}$$

Therefore, the variance converges with time to a constant value in the same rate as the distribution $p(x,t)$ does to the stationary state $P_1(x)$. In the singular case we obtain the simple expression

$$\sigma^2(t_0)=(\alpha+1)/\ln(t_0). \tag{32}$$

Obviously, both results, Eqs. (31) and (32), are valid only if t_0 is large. In contrast to the normal case, the variance (32) does not stabilize at a finite value; it falls to zero with t_0.

V. SUMMARY AND DISCUSSION

We have solved the CKE for KP, which is a simple, stepwise jumping process. The KP is Markovian and stationary, defined by a stationary transition probability. We have demonstrated that there are two kinds of time-independent probability distributions that the general solution converges to: the normal distribution, given as a smooth function of $Q(x)$ and $\nu(x)$, and a singular one, in the form of the δ function. The general solution of CKE for KP, involving arbitrary Q and ν, has been derived. The detailed analysis of that solution has been performed for the processes with algebraic $Q(x)$ and $\nu(x)$, defined on the finite interval of process values. In this case, it is possible to achieve simple asymptotic formulas for the CKE solutions. Those asymptotic expressions for the probability distribution are also algebraic functions of the process value x and, therefore, they correspond to some physically important phenomena. Moreover, it appears that, for the normal case, the probability distribution converges to the stationary state algebraically with time. On the other hand, the singular case exhibits even slower convergence to the limiting distribution, like $1/\ln t$. This particular time dependence of the asymptotic distribution appears to be generic for all processes leading to the singular stationary state among those we have considered.

Our findings demonstrate that, due to slow convergence, the stationary, time-independent state may not be reached in a reasonable time and in many practical applications the full solution must be taken into account. This conclusion is obvious for the singular case, but even for the normal one the speed of convergence can be arbitrarily slow if frequency $\nu(x)$ is characterized by a sufficiently high power index β. Moreover, we have demonstrated that also the choice of initial distribution is crucial for the convergence speed. Consequently, some KP would be effectively time dependent and, therefore, would mimic nonstationary processes, as soon as the observation time is not extremely long.

The fluctuation analysis confirms these conclusions. We have derived a general formula for the covariance and calculated the variance for both cases, Eqs. (14) and (19). In the normal case, the variance converges with t_0 to a constant value in the same way as the CKE solution does to the stationary limiting distribution. In the singular case, in turn, the variance does not stabilize with time at a finite value—this behavior is typical for stationary processes—but it falls to zero extremely slowly [see Eq. (32)].

The singular solution corresponds to the case of divergent normalization integral in Eq. (5): near $x=0$, the frequency $\nu(x)$ falls to zero fast, compared to $Q(x)$. Since the frequency is small there, process values close to $x=0$ are kept for a long time. Conversely, if $|x|$ is large, the process fluctuates rapidly, until eventually $|x|$ becomes small. As a result, the distribution is attracted to the $x=0$ point.

In fact, the normal case (14), for which $\alpha,\beta>0$, possesses the singular stationary solution $P_2(x)$ as well. However, the general solution (15) converges always to $P_1(x)$, unless the initial distribution $p(x,0)=\delta(x)$. Therefore, choosing some combination of the δ function and another arbitrary function as the initial condition, one obtains a combination of $P_1(x)$ and $P_2(x)$ as limiting stationary distribution at $t\rightarrow\infty$.

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APPENDIX

We want to find the conditional probability $P(x,t|x_0)$ that after time t the value of the process becomes x, provided that initially it was x_0. First, we consider the case that there was no jump at all. The probability $P_0(x,t|x_0)$ of such event follows directly from the definition of the KP:

$$P_0(x,t+\Delta t|x_0)=P_0(x,t|x_0)[1-\nu(x_0)\Delta t], \tag{A1}$$

where Δt is small. Therefore, we obtain the Poissonian distribution

$$P_0(x,t|x_0)=\exp[-\nu(x_0)t]\,\delta(x-x_0). \tag{A2}$$

Next we calculate the probability that there was just one jump during the time t. In order to calculate this probability, $P_1(x,t|x_0)$, we divide the time interval $[0,t]$ into n subintervals, such that $0=t_0<t_1<\cdots<t_n=t$, where n is large. The length of subsequent subintervals is $\Delta t_i=t_{i+1}-t_i$. Let us assume that the jump occurred in the ith subinterval. Therefore, to get the required probability we must multiply the probability of this event by the probability that there was no jump in the other subintervals and, finally, take a sum over all subintervals. The jump probability is given by the definition of KP: $\nu(x_0)Q(x)\Delta t_i$. Taking the limit $\Delta t_i\rightarrow 0$, we can change the sum over subintervals into the integral. Finally, we get

$$P_1(x,t|x_0)=\nu(x_0)Q(x)\int_0^t \exp[-\nu(x_0)s] \times\exp[-\nu(x)(t-s)]ds. \tag{A3}$$

Now we consider the general case of $k+1$ jumps: the process leads from x_0 to x through the sequence of values $x_1,\ldots,x_k$. The probability of that sequence is given by the following recurrence formula:

$$P_{k+1}(x,x_1,\ldots,x_k,t|x_0)=\nu(x_k)Q(x) \times\int_0^t P_k(x,x_1,\ldots,x_{k-1},s|x_0) \times\exp[(s-t)\nu(x)]ds. \tag{A4}$$

Taking into account all possible sequences of an arbitrary number of jumps, we get the expression for total conditional probability in the form

$$P(x,t|x_0)=\exp[-t\nu(x_0)]\,\delta(x-x_0) +\sum_{k=0}^{\infty}\int P_k(x,x_1,\ldots,x_{k-1},t|x_0)dx_1\cdots dx_k. \tag{A5}$$

The probability P_k in integral (A5) involves, according to Eq. (A4), the k-fold convolution. Therefore, it can be conveniently handled in terms of Laplace transforms. Transforming Eq. (A5) term by term, we have

$$\mathcal{L}_t[P(x,t|x_0)]=\frac{\delta(x-x_0)}{s+\nu(x_0)}+\sum_{k=0}^{\infty}\frac{\nu(x_0)}{s+\nu(x_0)}\frac{Q(x)}{s+\nu(x)} \times\left[\int\frac{\nu(x')Q(x')}{s+\nu(x')}dx'\right]^k, \tag{A6}$$

where the k-fold integral has been factorized. After some elementary algebra, we obtain the final formula for the Laplace transform of the required conditional probability:

$$\mathcal{L}_t[P(x,t|x_0)]=\frac{\delta(x-x_0)}{s+\nu(x_0)}+\frac{\nu(x_0)}{s+\nu(x_0)}\frac{Q(x)}{s+\nu(x)} \times\frac{1}{s\int Q(x')/[s+\nu(x')]dx'}. \tag{A7}$$

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